4/15/26

Last quiz!

Exam next Wednesday - practice exam will go up today.

Covers 3.1, 3.2, 3.3, 3.4, 5.1, 5.2, 5.3

Come early (5 min), leave late (5 min)

Summary 5.3.8

Let A be an $n \times n$ matrix and $T: \mathbb{R}^n \to \mathbb{R}^n$ the corresp lin. transf. TFAE

I. A is an orthogonal matrix

II. T preserves lengths: $\|T(\vec{x})\| = \|\vec{x}\| \quad \forall x \in \mathbb{R}^n$

III. The columns of A form an orthonormal basis of $\mathbb{R}^n$

IV $A^T A = I_n$

V $A^{-1} = A^T$

VI A preserves the dot product: $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$

VII A preserves angles.

Most of this we've talked about, but VI and VII are new. Let's talk a bit about them, namely why orthogonal $\Rightarrow$ VI $\Rightarrow$ VII

Assume A (hence T) is orthogonal.

Recall (last time)

$$\boxed{\underset{\text{dot prod}}{\vec{x} \cdot \vec{y}} = \underset{\text{matrix mult.}}{\vec{x}^T \vec{y}}} \quad (*)$$

Now:

$$T(\vec{x}) \cdot T(\vec{y}) = (A\vec{x}) \cdot (A\vec{y})$$

Since $A^T A = I_n$, $(A\vec{x})^T = x^T A^T = x^T A^{-1}$

↳ see part c of Th 5.3.9

$$\Rightarrow \underbrace{(A\vec{x}) \cdot (A\vec{y})}_{\text{dot prod}} = \underbrace{(A\vec{x})^T (A\vec{y})}_{\text{matrix mult.}}$$

$$= \left(\vec{x}^T A^T\right) A \vec{y}$$

$$= \vec{x}^T A^{-1} A \vec{y}$$

$$= \vec{x}^T \vec{y}$$

$$= \vec{x} \cdot \vec{y} \qquad (\text{by } (*))$$

<u>Corollary</u>

An orthogonal transformation preserves angles.

pf/

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|y\| \cos\theta$$

$$\|$$

$$T(\vec{x}) \cdot T(\vec{y}) = \|T(x)\| \, \|T(\vec{y})\| \cos\theta'$$

orthogonal $\Rightarrow$ $\|x\| = \|T(\vec{x})\|$

So $\cos\theta = \cos\theta'$, $\theta, \theta' \in [0, \pi)$

$$\Rightarrow \theta = \theta'$$

End of material for the exam.

§6.1 Determinants

- How do we compute them, what are they good for, and what properties do they have?

Recall $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$, and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0.$$

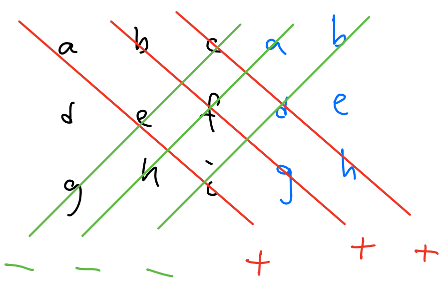
How about 3×3? We didn't talk about cross products, but just fyi:

if $A = [\vec{u} \mid \vec{v} \mid \vec{w}]$ then

$$\det A = \vec{u} \cdot (\vec{v} \times \vec{w}) \qquad \text{(don't memorize)}$$

Here's a simple way to compute it (Sarrus's rule)

To compute $\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ write it as



$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \cdots$$

Ex $\det\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \cdots = 0$